

THE AMALGAMATED FREE PRODUCT OF HYPERFINITE VON NEUMANN ALGEBRAS OVER FINITE DIMENSIONAL SUBALGEBRAS

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ABSTRACT. In this paper we describe the amalgamated free product of two hyperfinite von Neumann algebras over a finite dimensional subalgebra. In general the free product is of the form $H \oplus \bigoplus_{i=1}^N F_i$ where F_i are interpolated free group factors and H is a hyperfinite von Neumann algebra. We then show that the class of von Neumann algebras of this form is closed under taking amalgamated free products over finite dimensional subalgebras.

1. INTRODUCTION

The (reduced) amalgamated free products of C^* -algebras and of von Neumann algebras have proved to be useful constructions since first introduced by Voiculescu in [11] and used by Popa in [9]. For A and B finite von Neumann algebras, and D a unital subalgebra of both A and B , with trace preserving expectations E_D^A and E_D^B , we denote the amalgamated free product of A and B over D by $M = A *_D B$. In this paper we describe M in the case where A and B are hyperfinite and D is finite dimensional. This extends the results in [1], which examines the free product over the scalars of hyperfinite von Neumann algebras, and in [3], which examines the amalgamated free product of multimatrix algebras.

With Theorem 4.5 we also extend Theorem 4.4 in [5], which considered similar amalgamated free products but with additional restrictions on A and B and which was used by Kodiyalam and Sunder in [8] and in the related paper by Guionnet, Jones, and Shlyakhtenko in [6].

2. BASIC THEOREMS AND DEFINITIONS

We consider all von Neumann algebras to be equipped with a specified normal faithful tracial state. We will often write von Neumann algebras in the following way:

$$\mathcal{N} = \overset{p_1}{A_1} \oplus \overset{p_2}{A_2} \oplus \dots,$$

where p_i denotes the central support of A_i in \mathcal{N} . If a summand is a matrix algebra, we use the notation M_n to indicate that the trace of a minimal projection in M_n is t . As mentioned in the introduction, $A *_D B$ denotes the amalgamated free product of von Neumann algebras A and B over D with respect to a trace preserving conditional

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expectation onto D . When we refer to the expectation we mean the expectation onto D unless otherwise specified.

Our results are in terms of the interpolated free group factors that were introduced independently in [2] and by Radulescu in [10]. We shall denote these by $L(F_s)$, where $1 < s \leq \infty$. If s is an integer then it is the corresponding free group factor. As one would hope, these satisfy $L(F_s) * L(F_{s'}) = L(F_{s+s'})$, and we think of s as the (not necessarily integer) number of free generators of $L(F_s)$. When an interpolated free group factor is compressed by a projection of trace t or is dilated, we have

$$(L(F_s))_t = L(F_{1+(s-1)/t^2}),$$

$0 < t < \infty$. It is known that either $L(F_s) \cong L(F_{s'})$ for all s and s' or $L(F_s) \not\cong L(F_{s'})$ for all $s \neq s'$, though it remains open which of the two holds. This is the present state of knowledge about the isomorphism problem for free group factors.

If it turns out that the interpolated free group factors are not isomorphic to each other, then then it will be interesting to keep track of the number of generators whenever free group factors appear in constructions or results. For this purpose the (perhaps confusingly named) notion of *free dimension* was introduced in [1]. Ostensibly, the free dimension is a number assigned to certain finite von Neumann algebras endowed with normal, faithful, tracial states. For a finite von Neumann algebra M with given trace, we will denote the free dimension with respect to that trace by $\text{fdim}(M)$ (thus, the name of the trace is suppressed in the notation). The free dimension is defined by the following conventions:

- (1) For $M = L(F_s)$, $\text{fdim}(M) = s$.
- (2) For $M = \bigoplus_{i \in I} M_{n_i}$, $\text{fdim}(M) = 1 - \sum_{i \in I} t_i^2$.
- (3) For M a diffuse hyperfinite von Neumann algebra, $\text{fdim}(M) = 1$.
- (4) If

$$M = F_0 \oplus \bigoplus_{i \in I} L(F_{s_i})^{p_i} \oplus \bigoplus_{j \in J} M_{n_j}^{t_j},$$

where F_0 is a diffuse hyperfinite von Neumann algebra, or zero, and I and J may be empty, then,

$$\text{fdim}(M) = 1 + \left(\sum_{i \in I} \tau(p_i)^2 (s_i - 1) \right) - \sum_{j \in J} t_j^2.$$

In fact, as is apparent from item (1), we do not know if $\text{fdim}(M)$ is in all cases well defined. There are several ways around this problem. One is to say, for the purpose of interpreting results, we only care about the free dimension if the free group factors are not isomorphic to each other, in which case it is well defined. Another more cumbersome but mathematically correct approach is to speak only of free dimension for generating sets of finite von Neumann algebras. This approach was taken in [5] and earlier in [4], and corresponds well to results about Voiculescu's free entropy dimension (which is intrinsically a different notion, see [12], [13] and [7]). See section 3 of [5] and section 1 of [4] for more discussion. A third approach is to speak only of "the conjectured free dimension" which would mean that we conjecture that the free group factors are nonisomorphic, (in which case the free dimension has meaning);

however, we are not ready to conjecture nonisomorphism of free group factors. In this article, we will use the notation $\text{fdim}(M) = x$ as shorthand for (but without actually phrasing) the more cumbersome “has a generating set of free dimension x .”

In [1] the concept of a *standard embedding* from one interpolated free group factor into another is introduced. We will rely heavily on this concept, in particular the following properties, shown in [1],

- (1) For $A = L(F_s)$ and $B = L(F_{s'})$, $s < s'$, then for $\phi : A \rightarrow B$ and projection $p \in A$, ϕ is standard if and only if $\phi|_{pAp} \rightarrow \phi(p)B\phi(p)$ is standard.
- (2) The inclusion $A \rightarrow A * B$ is standard if A is an interpolated free group factor and B is an interpolated free group factor, $L(\mathbb{Z})$, or a finite dimensional algebra other than \mathbb{C} .
- (3) The composition of standard embeddings is standard.
- (4) For $A_n = L(F_{s_n})$, with $s_n < s_{n'}$ if $n < n'$, and $\phi_n : A_n \rightarrow A_{n+1}$ a sequence of standard embeddings, then the inductive limit of the A_n with the inclusions ϕ_n is L_{F_s} where $s = \lim_{n \rightarrow \infty} s_n$.

3. USEFUL LEMMAS

Lemma 3.1. *Let \mathcal{M} be a hyperfinite von Neumann algebra with separable predual and D be a finite dimensional abelian subalgebra of it. Then there exists a chain of finite dimensional subalgebras in \mathcal{M} containing D whose union is dense in \mathcal{M} .*

Proof. \mathcal{M} can be split as the direct sum of a type I part and a type II part, each of which can be approximated separately, so we can deal with these cases individually.

If \mathcal{M} is type I then write it as $\bigoplus_{i \in I} M_{m_i} \otimes L^{\infty}(X, \mu_i)^{p_i}$, for some finite measures μ_i and countable set I . We may identify each $M_{m_i} \otimes L^{\infty}(X, \mu_i)$ with $M_{m_i}(L^{\infty}(X, \mu_i))$ in such a way that all elements of D are identified with diagonal matrices. Write $D = \bigoplus_{k \in I_D} \mathbb{C}^{p_k^D}_{t_k^D}$ with some finite index set I_D .

Set $A_0 = D$, and let $I_n \subseteq I$ consist of the first n elements of I (if I is finite then $I_n = I$ for all $n > |I|$). Set $P_{0,i} = \{X\}$ for all $i \in I$ and inductively choose $P_{n,i}$ to be a partition of X so that: a) the measure μ_i of every set in $P_{n,i}$ is less than $1/2^n$ or is composed of a single atom, b) the partition $P_{n,i}$ is compatible with D for M_{m_i} , and c) $P_{n,i}$ refines $P_{n-1,i}$. By $P_{n,i}$ compatible with M_{m_i} we mean that for each $e_{jj} \in M_{m_i}$ and $S_m \in P_{n,i}$, then $\chi_{S_m} e_{jj} \leq p_k^D$ for some $k \in I_D$. Then set A_n to be

$$\left(\bigoplus_{i \in I_n} M_{m_i} \otimes \ell^{\infty}(P_{n,i}, \mu_i)^{p_i} \right) \oplus \mathbb{C}^{k_n}.$$

In the above the \mathbb{C}^{k_n} is the span of $\{p_k^D - p_k^D (\sum_{i \in I_n} p_i), k \in I_D\}$. So we have constructed $D = A_0 \subseteq A_1 \subseteq \dots$ so that $\cup A_i$ is dense in \mathcal{M} .

For the type II case we assume $D = \mathbb{C} \oplus \mathbb{C}$, and from there the general case can be done inductively. Let p_1 and p_2 be the minimal projections in D , $p_1 + p_2 = 1$. We write $\mathcal{M} = L^{\infty}(X, \mu) \otimes R$ where R is the hyperfinite II_1 factor and μ is a finite measure. We choose a representation of R as the closure of $\cup_k M_{2^k}$, and denote the

standard basis elements of M_{2^k} as $e_{i,j}^{(2^k)}$. Use the inclusion of M_{2^k} into $M_{2^{k+1}}$ which takes $e_{i,j}^{(2^k)}$ to $e_{2i-1,2j-1}^{(2^{k+1})} + e_{2i,2j}^{(2^{k+1})}$.

We know that projections are equivalent exactly when they have the same centre-valued trace. Let $f \in L^\infty(\mu)$ be the centre-valued trace of p_1 . Note f takes values only in $[0, 1]$. We can construct a projection which has centre-valued trace f in the following way:

$$p'_1 = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} e_{(2i-1)(2i-1)}^{(2^k)} \otimes \chi_{f^{-1}((\frac{2i-1}{2^k}, \frac{2i}{2^k}])}$$

where χ is the characteristic function. To see how that works, note that any of the partial sums:

$$p'_{1,N} = \sum_{k=1}^N \sum_{i=1}^{2^{k-1}} e_{(2i-1)(2i-1)}^{(2^k)} \otimes \chi_{f^{-1}((\frac{2i-1}{2^k}, \frac{2i}{2^k}])}$$

is a member of $L^\infty \otimes M_{2^N}$. Considering $p'_{1,N}$ as an M_{2^N} valued function on X , note that in each region where f takes values between $i/2^N$ and $(i+1)/2^N$, $p'_{1,N}$ is the matrix with 1s down the diagonal for the first i entries, and zeros for the rest. This means the centre-valued trace of $p'_{1,N}$ is the simple function f_N taking only values of the form $i/2^N$ which approximates f from below. Thus $\|f_N - f\| \leq 1/2^N$, and so p'_1 has the desired centre-valued trace, and is thus equivalent to p_1 . So without loss of generality, by modifying our choice of dense subalgebra we can assume p_1 is of this form.

Start by defining the partition P_1 of X as $P_1 = \{S_{1,1}, S_{1,2}\}$ where $S_{1,1} = f^{-1}([0, 1/2])$ and $S_{1,2} = f^{-1}((1/2, 1])$, and define $\phi_{1,n}$ to be the characteristic function of $S_{1,n}$ times the identity. Next define A_1 to be the algebra generated by $\{e_{1,1}^{(2)}\phi_{1,2}, e_{2,2}^{(2)}\phi_{1,1}, p_1 - e_{1,1}^{(2)}\phi_{1,2}, p_2 - e_{2,2}^{(2)}\phi_{1,1}\}$. Note these are four orthogonal projections, so they generate \mathbb{C}^4 which is clearly finite dimensional. It is also clear that p_1 and $1 - p_1$ are in this algebra, thus it contains D .

Next inductively construct partitions $P_k = \{S_{k,1}, \dots, S_{k,\ell}\}$, satisfying the following conditions: a) For any $S_{k,n} \in P_k$ either its measure is less than $1/2^k$ or it is composed of a single atom, b) the range of f on each $S_{k,n} \in P_k$ is contained in $(i/2^k, (i+1)/2^k]$ for some integer i , and c) P_k refines P_{k-1} .

As before, we define $\phi_{k,n}$ to be the characteristic function of $S_{k,n}$. Define a function r_k on P_k , so that $r_k(S_{k,n}) = i$ where $f(S_{k,n}) \subseteq (i/2^k, (i+1)/2^k]$. We set A_k to be the algebra generated by:

$$\left\{ e_{i,j}^{(2^k)} \phi_{k,n}, \left(p_1 \phi_{k,n} - \sum_{m=1}^{r_k(S_{k,n})} e_{m,m}^{(2^k)} \phi_{k,n} \right), \right. \\ \left. \left(p_2 \phi_{k,n} - \sum_{m=r_k(S_{k,n})+2}^{2^k} e_{m,m}^{(2^k)} \phi_{k,n} \right) \mid S_{k,n} \in P_k, i \neq r_k(S_{k,n}), j \neq r_k(S_{k,n}) \right\}.$$

By our representation of p_1 , for $S_{k,n} \in P_k$ if $i < r_k(S_{k,n})+1$ then $e_{i,i}^{(2^k)} \phi_{k,n} \leq p_1$ and if $i > r_k(S_{k,n})+1$, $e_{i,i}^{(2^k)} \phi_{k+1,n} \leq p_2$, thus the algebra we get is $\bigoplus_{S_{k,n} \in P_k} (M_{2^k-1} \oplus \mathbb{C} \oplus \mathbb{C})$.

Since $p_1 \phi_{k,n} - \sum_{i=1}^{r_k(S_{k,n})} e_{i,i}^{(2^k)} \phi_{k,n}$ is in A_k , and each of the elements of the sum is also in A_k , $p_1 \phi_{k,n} \in A_k$ for each $S_{k,n} \in P_k$, and thus p_1 is, and by the same argument so is p_2 thus $D \subseteq A_k$.

To check that A_{k-1} is a subalgebra of A_k , take any of the $e_{i,j}^{(2^{k-1})} \phi_{k-1,n}$ with $i, j \neq r_{k-1}(S_{k-1,n})$, then

$$\sum_{n', S_{k,n'} \subseteq S_{k-1,n}} \left(e_{2i-1, 2j-1}^{(2^k)} \phi_{k,n'} + e_{2i, 2j}^{(2^k)} \phi_{k,n'} \right) = e_{i,j}^{(2^{k-1})} \phi_{k-1,n}.$$

For $p_1 \phi_{k-1,n} - \sum_{i=1}^{r_{k-1}(S_{k-1,n})} e_{i,i}^{(2^{k-1})} \phi_{k-1,n}$, we have already established that $p_1 \phi_{k,n'}$ for any n' is in A_k and so since $p_1 \phi_{k-1,n} = \sum_{n', S_{k,n'} \subseteq S_{k-1,n}} p_1 \phi_{k,n'}$ it is in A_k . Since all the terms in the sum are also in A_k , then $p_1 \phi_{k-1,n} - \sum_{i=1}^{r_{k-1}(S_{k-1,n})} e_{i,i}^{(2^{k-1})} \phi_{k-1,n}$ must be too. Thus $A_{k-1} \subseteq A_k$.

Furthermore we see this is dense in \mathcal{M} since our restriction on the measure of $S_{k,n}$ ensures ever greater refinement in our approximation of $L^\infty[\mu]$, and R is approximated by $M_{2^{k+1}}$ with only one row and one column missing, whose contribution goes to zero as k goes to infinity. □

Notation: We denote $\Lambda^0(X, Y)$ to be the set of all non-empty alternating words of elements of X and Y . For an algebra X with a trace, we use X^0 to denote the traceless elements of X and if it has an expectation onto D , we use X^{00} to denote the expectationless elements of X .

Lemma 3.2. *Let $\mathcal{N} = (M_m \oplus M_{n-m} \oplus B) *_D C$ and $\mathcal{M} = (M_n \oplus B) *_D C$, where B, C are von Neumann algebras with finite trace and D is finite dimensional and abelian. We embed \mathcal{N} in \mathcal{M} by mapping M_m and M_{n-m} as blocks on the diagonal of M_n , and B and C by the identity. Assume there exists a factor \mathcal{F} with $M_m \oplus M_{n-m} \subseteq \mathcal{F} \subseteq \mathcal{N}$. Let p_i^D be a minimal projection in D . Then for any minimal projection $p \in M_m$ such that $p \leq p_i^D$, $p\mathcal{N}p * L(\mathbb{Z}) \cong p\mathcal{M}p$*

Proof. Denote $A = M_m \oplus M_{n-m} \oplus B$ and $A' = M_n \oplus B$. We represent the matrices in such a way that D embeds diagonally and $p = e_{11}$

Note $\mathcal{M} = v\mathcal{N}(\mathcal{N} \cup e_{1n})$. Now since \mathcal{F} is a factor we can find a partial isometry $v \in \mathcal{F} \subseteq \mathcal{N}$ such that $v^*v = e_{11}$ and $vv^* = e_{nn}$. Let $u = v^*e_{n1}$, and thus $uu^* = u^*u = e_{11}$.

Since u and \mathcal{N} generate \mathcal{M} we know that u and $e_{11}\mathcal{N}e_{11}$ generate $e_{11}\mathcal{M}e_{11}$. We claim that u and $e_{11}\mathcal{N}e_{11}$ are $*$ -free with respect to the trace. We must show that any element of $\Lambda^0(\{u^k | k \in \mathbb{Z} \setminus \{0\}\}, (e_{11}\mathcal{N}e_{11})^0)$ has zero trace.

By breaking up u into v^*e_{n1} , and noting $v \in \mathcal{N}$, and $v = e_{nn}ve_{11}$ we see that

$$\Lambda^0(\{u^k | k \in \mathbb{Z} \setminus \{0\}\}, (e_{11}\mathcal{N}e_{11})^0) \subseteq \Lambda^0(\{e_{1n}, e_{n1}\}, \{(e_{11}\mathcal{N}e_{11})^0, e_{11}\mathcal{N}e_{nn}, e_{nn}\mathcal{N}e_{11}, (e_{nn}\mathcal{N}e_{nn})^0\}).$$

We can write $x - E_D^\mathcal{N}(x)$ for any $x \in \mathcal{N}$ as the SOT limit of a bounded sequence in the span of $\Lambda^0(A^{00}, C^{00})$. Then since non-zero elements of $e_{11}Ae_{11}$ or $e_{nn}Ae_{nn}$ have

non-zero trace, we can write elements of $(e_{11}\mathcal{N}e_{11})^0$ and $(e_{nn}\mathcal{N}e_{nn})^0$ as SOT limits of bounded sequences in the span of $\Lambda^0(A^{00}, C^{00}) \setminus A^{00}$ (thus the expectation is zero also). Similarly note that since $e_{11}Ae_{nn} = 0 = e_{nn}Ae_{11}$, elements of $e_{11}\mathcal{N}e_{nn}$ and $e_{nn}\mathcal{N}e_{11}$ can be written in the same way.

Thus we can approximate any element of

$$\Lambda^0(\{e_{1n}, e_{n1}\}, \{(e_{11}\mathcal{N}e_{11})^0, e_{11}\mathcal{N}e_{nn}, e_{nn}\mathcal{N}e_{11}, (e_{nn}\mathcal{N}e_{nn})^0\})$$

by bounded sequences in the span of $\Lambda^0(\{e_{n1}, e_{1n}\}, \text{span}(\Lambda^0(A^{00}, C^{00}) \setminus A^{00}))$. Now note that $E_D(Ae_{n1}A) = E_D(Ae_{1n}A) = 0$. Thus

$$\Lambda^0(\{e_{n1}, e_{1n}\}, \text{span}(\Lambda^0(A^{00}, C^{00}) \setminus A)) \subseteq \text{span}(\Lambda^0(A^{00}, C^{00})).$$

By freeness, elements of $\Lambda^0(A^{00}, C^{00})$ have expectation zero, and thus trace zero, and so u and $e_{11}\mathcal{N}e_{11}$ are $*$ -free and u is a Haar unitary.

Thus $e_{11}\mathcal{N}e_{11} * L(Z) \cong e_{11}\mathcal{M}e_{11}$.

□

Lemma 3.3. *Let $\mathcal{M} = ((M_n \otimes A) \oplus B) *_D C$ and $\mathcal{N} = (M_n \oplus B) *_D C$ for A, B , and C von Neumann algebras with finite trace, and D a finite dimensional abelian von Neumann algebra, where C, B, M_n have expectations onto D and where $E_D^{M_n \otimes A} = E_D^{M_n} \otimes \tau_A$. Let p be a minimal projection in M_n which sits under a minimal projection in D . Then $p\mathcal{N}p * A \cong p\mathcal{M}p$*

Proof. Since pA and \mathcal{N} generate \mathcal{M} we see that $p\mathcal{N}p$ and A embedded as pA generate $p\mathcal{M}p$. Thus we need only show that these two algebras are $*$ -free.

As in the previous proof, we see that traceless elements of $p\mathcal{N}p$ are expectationless and traceless elements of $p(M_n \oplus B)p$ are zero, so elements of $p\mathcal{N}p$ can be approximated by $\text{span}(\Lambda^0((M_n \oplus B)^{00}, C^{00}) \setminus (M_n \oplus B)^{00})$. It is easy to see that traceless elements of pA are also expectationless.

Any traceless element of pA multiplied on both left and right by elements of M_n is expectationless (since each entry has zero trace in A). Thus any element of $\Lambda^0((pA)^0, (p\mathcal{N}p)^0)$ can be approximated by words in the span of $\Lambda^0(((M_n \otimes A) \oplus B)^{00}, C^{00})$, and thus expectationless and traceless, and thus pA and $p\mathcal{M}p$ are $*$ -free.

□

Definition 3.4. We call an embedding $\phi : \mathcal{M} \rightarrow \mathcal{N}$ a *simple step* if it follows one of the two following patterns:

- (1) $\mathcal{N} = \overset{p_1}{A} \oplus \overset{q}{B}$, $\mathcal{M} = \overset{p_2}{A} \oplus \overset{p_3}{A} \oplus \overset{q}{B}$, with $p_1 = p_2 + p_3$, and $\phi(a, b) = (a, a, b)$.
- (2) $\mathcal{N} = \overset{p}{M_n} \oplus \overset{p}{M_m} \oplus \overset{p}{B}$, $\mathcal{M} = \overset{p}{M_{n+m}} \oplus \overset{p}{B}$ where $\phi(x, y, b) = \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, b \right)$.

Lemma 3.5. *If we have two finite dimensional von Neumann algebras \mathcal{N} and \mathcal{M} and a trace preserving embedding $\phi : \mathcal{N} \rightarrow \mathcal{M}$, then ϕ can be written as the composition of as sequence of simple steps.*

Proof. Since \mathcal{N} and \mathcal{M} are both finite dimensional we can write them as direct sums of matrix algebras, $\mathcal{N} = \overset{p_1}{M_{n_1}} \oplus \cdots \oplus \overset{p_s}{M_{n_s}}$ and $\mathcal{M} = \overset{q_1}{M_{m_1}} \oplus \cdots \oplus \overset{q_t}{M_{m_t}}$. Then consider the Bratteli diagram for this embedding, $G = (V, E)$, $V = \{p_i, q_j \mid 1 \leq i \leq s, 1 \leq j \leq t\}$.

For each edge linking p_i to q_j we decorate it as usual with the partial multiplicity of the embedding, $a_{i,k}$, in addition we decorate it with a number $\alpha_{i,j}$, which is the trace of $p_i q_j$.

Then to break this map into simple steps, we start by making copies of the M_{n_j} by using the first type of simple step, so that for each edge $e_{i,j}$ there are $a_{i,j}$ copies of M_{n_j} , each with trace $\alpha_{i,j}/a_{i,j}$. This leaves us with a multimatrix algebra, where each matrix algebra is associated with an edge $e_{i,j}$ (which is associated to $a_{i,j}$ matrix algebras). Then for every $1 \leq j \leq t$, we use the second type of simple step to join together all the matrix algebras associated to edges ending in j .

□

Definition 3.6. We define the graph G_D^A of a hyperfinite von Neumann Algebra A and multimatrix subalgebra $D = \oplus_{i \in I_D} \overset{p_i^D}{M_{n_i}}$ as follows. Our vertex set is I_D . Vertices $i, j \in I_D$ are connected by an edge if $p_j A p_i \neq 0$.

We also use $G_D^{A,B}$ to denote the union of the graphs G_D^A and G_D^B , where B is another von Neumann algebra with D as a subalgebra.

Remark: Note the graph as defined for multimatrix algebras in [3] is connected if and only if the graph in Definition 3.6 is connected, however it is not the same graph.

4. MAIN RESULTS

Proposition 4.1. *Let A and B be hyperfinite von Neumann algebras with finite trace, and let D be a finite dimensional subalgebra of both, so that the graph $G_D^{A,B}$ is connected, and so that no minimal projection in D is also minimal in A or B . Then $A *_D B = F \oplus_{r=1}^R M_{n_r}$ where F is either an interpolated free group factor or a diffuse type I hyperfinite algebra. Furthermore $\text{fdim}(A *_D B) = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D)$.*

Proof. The proof of Lemma 5.2 from [3] shows that we may assume without loss of generality that D is abelian.

Write $A = \overset{p_d^A}{A_d} \oplus \overset{p_a^A}{A_a}$ and $B = \overset{p_d^B}{B_d} \oplus \overset{p_a^B}{B_a}$, where A_a and B_a are the type I atomic parts, and A_d and B_d the diffuse parts. Write $D = \oplus_{k \in I_D} \overset{p_k^D}{t_k^D} \mathbb{C}$, $A_a = \oplus_{\ell \in I_{A_a}} \overset{p_\ell}{t_\ell} M_{n_\ell}$, and $B_a = \oplus_{\ell \in I_{B_a}} \overset{p_\ell}{t_\ell} M_{n_\ell}$, with I_D , I_{A_a} , and I_{B_a} disjoint.

Now using Lemma 3.1 we can construct a sequence $A_d(i)$ of finite dimensional subalgebras of A_d containing $p_d^A D p_d^A$ and whose inductive limit is A_d . Denote $A_d(i) = \oplus_{\ell \in I_{A_d(i)}} \overset{p_\ell}{t_\ell} M_{n_\ell}$, and $B_d(j) = \oplus_{\ell \in I_{B_d(j)}} \overset{p_\ell}{t_\ell} M_{n_\ell}$. Note I_D , $I_{A_d(i)}$, and $I_{B_d(j)}$, are finite, but I_{A_a} and I_{B_a} may not be.

Also, since A_d has no minimal projections, we can choose the $A_d(i)$ in such a way that the trace of the any minimal projection $p \in A_d(i)$, $p \leq p_k^D$ is less than $t_k^D - \tau(p')$ for any minimal projection $p' \in B$, $p' \leq p_k^D$. Note we can do this, since we know the minimal projections $p' \in B$, $p' \leq p_k^D$ have traces adding to t_k^D , and they must be strictly less than t_k^D . We can use Lemma 3.5 to make sure the steps between $A_d(i)$ and $A_d(i+1)$ are all simple steps. We construct $B_d(j)$ in the same way.

Denote $A(i) = A_d(i) \oplus A_a$ and $B(j) = B_d(j) \oplus B_a$. Note $A(i) = \oplus_{\ell \in I_{A(i)}} M_{n_\ell}^{t_\ell}$ where $I_{A(i)} = I_{A_d(i)} \cup I_{A_a}$. Start our sequences far enough along that $G_D^{A(i), B(j)} = G_D^{A, B}$.

First, we claim that for any i, j

$$M(i, j) = A(i) *_D B(j) = F_{i,j}^{p_0^M} \oplus \bigoplus_{r=1}^R M_{n_r}^{p_r^M},$$

where $F_{i,j}$ is either an interpolated free group factor or a diffuse type I hyperfinite algebra. We will also show that only the $F_{i,j}$ depends on i and j . Then we shall show that $F_{i,j} \rightarrow F_{i+1,j}$ is standard.

We begin by examining the construction in Theorem 5.1 in [3]. For this we construct $N_{i,j}(k) = vN(p_k^D A(i) p_k^D \cup p_k^D B(j) p_k^D)$ for each k . This is of the form $F_k \oplus \bigoplus_{y \in Y(k)} M_{n_y}(\mathbb{C})$, where F_k is either an interpolated free group factor or a diffuse type I hyperfinite algebra. Here each element of $y \in Y(k)$ corresponds to a pair (ℓ, ℓ') , $\ell \in I_{A(i)}$, $\ell' \in I_{B(j)}$, representing a pair of matrix algebras in $A(i)$ and $B(j)$, so that $t_\ell/\lambda_{\ell,k} + t_{\ell'}^B/\lambda_{\ell',k} > t_k^D$, where $\lambda_{\ell,k}$ is the partial multiplicity of p_k^D in M_{n_ℓ} (i.e. $M_{\lambda_{\ell,k}} \cong p_k^D M_{n_\ell} p_k^D$), and the same for $\lambda_{\ell',k}$. Importantly, because of our choice of $A_d(i)$ and $B_d(j)$, if $t_\ell/\lambda_{\ell,k} + t_{\ell'}^B/\lambda_{\ell',k} > t_k^D$ then $\ell \in I_{A_a}$ and $\ell' \in I_{B_a}$.

Define a *connector* v to be a partial isometry in $A(i)$ (or $B(j)$) so that vv^* and v^*v are minimal in $A(i)$ (or $B(j)$) and so $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$ for some $k, k' \in I_D$.

All further construction affecting the matrix algebras of $M(i, j)$ is based on those $Y(k)$ s and connectors between them. Thus, these are entirely determined by the A_a and B_a , and thus remain unchanged from $M(i, j)$ to $M(i+1, j)$.

Now we show that the various summand factors F_k are linked by connectors. To do this we show that for any connector v in $A(i)$ or $B(j)$ with $vv^* \leq p_k^D$, vv^* is not orthogonal to F_k . Without loss of generality, assume $v \in A(i)$. Then $vv^* \in N_{i,j}(k)$, and is a minimal projection in $p_k^D A(i) p_k^D$. Using Theorem 3.2 in [3], we can determine the traces of the matrix factor summands of $N_{i,j}(k)$ which are not orthogonal to vv^* . Let $t = \tau(vv^*)$, and λ the partial multiplicity of p_k^D in the factor summand of $A(i)$ containing vv^* . Then the matrix factor summands of $N_{i,j}(k)$ which are not orthogonal to vv^* correspond to those factor summands $M_{\lambda_{\ell,k}}^{t_{\ell,k}}$ of $p_k^D B(j) p_k^D$ with

$t/\lambda + t_{\ell,k}/\lambda_{\ell,k} > t_k^D$. Note this can only happen if at least one of λ or $\lambda_{\ell,k}$ equals 1. Let $L \subseteq I_{B(j)}$ be the set of ℓ satisfying this inequality. It is easy to check that L is finite.

First let us assume $\lambda = 1$. Then for each $\ell \in L$ we have a matrix algebra

$$M_{\lambda_{\ell,k}}^{t_{\ell,k}}. \text{ Thus the total trace of these is } \lambda_{\ell,k}(t + \frac{t_{\ell,k}}{\lambda_{\ell,k}} - t_k^D)$$

$$\sum_{\ell \in L} \lambda_{\ell,k}^2 (t + \frac{t_{\ell,k}}{\lambda_{\ell,k}} - t_k^D) \leq t \left(\sum_{\ell \in L} \lambda_{\ell,k}^2 \right) + t_k^D - t_k^D \left(\sum_{\ell \in L} \lambda_{\ell,k}^2 \right) = t - (t_k^D - t) \left(\sum_{\ell \in L} \lambda_{\ell,k}^2 - 1 \right),$$

since $\lambda_{\ell,k} t_{\lambda_{\ell,k}} = \tau(p_k^D I_{M_\ell} p_k^D)$, and $p_k^D I_{M_\ell} p_k^D \leq p_k^D$. Now since $t_k^D > t$, unless $|L| = 1$ and $\lambda_{\ell,k} = 1$ this is strictly less than t . If $\lambda_{\ell,k} = 1$, and $|L| = 1$, the first inequality holds strictly, since otherwise $\tau(p_k^D) = \tau(p_k^D I_{M_\ell} p_k^D)$ and this is minimal in $B(j)$, thus p_k^D is minimal in B , contradicting our assumption. This shows the total trace of the

matrix factor summands of $N_{i,j}(k)$ not orthogonal to vv^* is strictly less than that of vv^* , and thus F_k is not orthogonal to vv^* .

If instead $\lambda > 1$ and thus $\lambda_{\ell,k} = 1$ for each $\ell \in L$. Let $N = |L|$, then for each such $\ell \in L$ we have a matrix algebra M_λ . Thus the total trace of these is:

$$\lambda(\frac{t}{\lambda} + t_\ell - t_k^D)$$

$$\sum_{\ell \in L} \lambda^2(\frac{t}{\lambda} + t_\ell - t_k^D) \leq tN\lambda + \lambda^2 t_k^D - \lambda^2 N t_k^D = t\lambda - \lambda(N-1)(\lambda t_k - t),$$

which is then strictly less than $t\lambda$ if $N > 1$. As before, if $N = 1$, then the first inequality holds strictly, otherwise p_k^D would be minimal in B . In this case, since vv^* is a minimal projection in M_λ as a factor summand of $N_{i,j}(k)$, we can form p_1, \dots, p_λ , minimal projections in $N_{i,j}(k)$ which are all equivalent to vv^* in $N_{i,j}(k)$ and mutually orthogonal. Thus the trace of their sum is $t\lambda$, which we have established is strictly greater than the total trace of all the matrix factor summands not orthogonal to them, and so they must not be orthogonal to F_k , and so vv^* is not either.

The construction of $M(i, j)$ in Theorem 5.1 in [3], proceeds by taking $N_{i,j} = \oplus_{k \in I_D} N_{i,j}(k)$, and adjoining connectors. Let S denote the set of connectors added, and let $N_{i,j}(S') = vN(N_{i,j} \cup S')$, for any $S' \subseteq S$. Thus $M(i, j) = N_{i,j}(S)$. Let S_v be the set of connectors added before $v \in S$. From the above, for any $v \in S$ we know that neither vv^* nor v^*v are minimal in $N_{i,j}$, thus when move from $N_{i,j}(S_v)$ to $N_{i,j}(S_v \cup \{v\})$, we apply Lemma 4.2 in [3], to determine $\bar{v}N_{i,j}(S_v \cup \{v\})\bar{v}$ where $\bar{v} = vv^* + v^*v$. This shows that it contains exactly one free group factor (it can't be a hyperfinite, since we know $\dim \bar{v}N_{i,j}(S_v)\bar{v} = \infty > 4$). Thus for any $v \in S$, $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$ for $k, k' \in I_D$, then F_k and $F_{k'}$ are contained in some F which is an interpolated free group factor summand of $M(i, j)$ (noting if F_k is hyperfinite, then

$$p_k^D A(i) p_k^D = \mathbb{C} \oplus_{p_1}^{p_k^D - p_1} \mathbb{C} \text{ and } p_k^D B(i) p_k^D = \mathbb{C} \oplus_{p_2}^{p_k^D - p_2} \mathbb{C}, \text{ and thus } vv^* \in \{p_1, p_2, 1-p_1, 1-p_2\}.$$

By Theorem 1.1 in [1] all of these have central support which contains I_{F_k} . Then since the graph $G_D^{A(i), B(j)}$ is connected, there is exactly one free group factor summand of $M(i, j)$, unless, $A = \mathbb{C} \oplus \mathbb{C}$, $B = \mathbb{C} \oplus \mathbb{C}$ and $D = \mathbb{C}$, in which case it is a type I diffuse hyperfinite algebra.

Next we show that the inclusion of $A_d(i)$ into $A_d(i+1)$ is standard. Since it is a simple step it is one of two types. First it could make two copies of a matrix algebra M_{n_ℓ} in $A_d(i)$ contained in the free group factor summand $F_{i,j}$ in $M(i, j)$. Lemma 3.3 shows that $F_{i,j} \rightarrow F_{i+1,j}$ is a standard embedding.

Alternately it could be the addition of a connector, connecting $M_{n_\ell}, M_{n_{\ell'}} \in A_d(i)$, giving us $M_{\ell''} \in A_d(i+1)$. Since M_{n_ℓ} and $M_{n_{\ell'}}$ are contained in $F_{i,j}$, we can apply Lemma 3.2 to show $F_{i,j} \rightarrow F_{i+1,j}$ is a standard embedding.

By Theorem 5.1 in [3] we know that at each stage the free dimension of $M(i, j)$ is the sum of the free dimensions of $A(i)$ and $B(j)$ minus that of D . By Proposition 4.3 in [1] we know that since the embeddings are standard, the free dimension of the inductive limit of the $M(i, j)$ is the limit of the free dimensions of the $M(i, j)$. Thus the inductive limit, $A *_D B$ has free dimension of the sum equal to that of A and B minus that of D .

Note that since the matrix portion remain constant and thus can be computed by computing an earlier stage in the chain, using Theorem 5.1 in [3]. \square

Theorem 4.2. *Let A and B be hyperfinite von Neumann Algebras, and D be a finite dimensional subalgebra of both, with $G_D^{A,B}$ connected. Then $A *_D B$ is the direct sum of a finite number of interpolated free group factors and possibly a hyperfinite von Neumann algebra. Furthermore $\text{fdim}(A *_D B) = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D)$.*

Proof. As before we assume D is abelian and set $A = A_a \oplus A_d$ and $B = B_a \oplus B_d$. In choosing our sequence the only difference from the situation in Proposition 4.1 is that when we ensure the minimal projections $p \in A_d(i)$, $p < p_k^D$ have smaller trace than $t_k^D - \tau(p')$, for all minimal projections $p' \in B$, $p' < p_k^D$, we are excluding the cases where $p' = p_k^D$ (in the context of prop 4.1 these did not exist). In the case where p_k^D is minimal in B we instead require that $\tau(p) \leq t_k^D/4$ for all $p \in A, p \leq p_k^D$.

Unfortunately it is no longer true that we must have only one free group factor summand, nor that all of $A_d(i)$ and $B_d(j)$ must be contained in interpolated free group factors. Nor is it true that all elements of $p_k^D A_d(i) p_k^D$ for some k must be in the same interpolated free group factor.

Instead for each free group factor summand F in $M(i, j)$ which is not orthogonal to both A_d and B_d , we shall associate a non-empty subset $Q_F \subseteq I_D$, so that $Q_F \cap Q_{F'} = \emptyset$ if $F \neq F'$. Furthermore if $k \in Q_F$ then we will have $p_k^D F' p_k^D = 0$ for all $F' \neq F$.

Each $k \in I_D$ can be assigned to a Q_F in one of two ways, or not assigned at all. *Method I* is used if p_k^D is not minimal in either A or B . Then we have established that there is one interpolated free group factor summand F of $M(i, j)$ which contains $p_k^D A_d(i) p_k^D$ and $p_k^D B_d(j) p_k^D$, we assign k to Q_F .

Suppose on the contrary, for $k \in I_D$ that p_k^D is minimal in either A or B , and without loss of generality assume it is B . Additionally assume p_k^D is not orthogonal to A_d .

As in the previous proof, and the proof of Theorem 5.1 in [3], we construct $M(i, j)$ with $N_{i,j}(k) = vN(p_k^D A_d(i) p_k^D \cup p_k^D B_d(j) p_k^D)$, and then define $N_{i,j} = \bigoplus_{k \in I_D} N_{i,j}(k)$. Set $N_{i,j}(S) = vN(N_{i,j} \cup S)$ where S is a set of connectors.

Then there is a set $S \subseteq (A(i) \cup B(j))$ of connectors so that $M(i, j) = N_{i,j}(S)$. If there is a connector v such that either v^*v or vv^* equals p_k^D , then $v \in B_a$, since p_k^D is minimal in B and not in $A(i)$. Without loss of generality we can assume that there is at most one $v \in S$ so that vv^* or v^*v is p_k^D (without loss of generality assume vv^*), since if there is another w so that $ww^* = p_k^D$ we can replace it with v^*w .

We use *Method II* when there is such a v and v^*v is not minimal and central in $\bar{q}N_{i,j}(S_0)\bar{q}$ where $\bar{q} = vv^* + v^*v$ and $S_0 = S \setminus \{v\}$.

First let us show that this does not depend on the choice of v . Assume there exists v and v' , where $vv^* = v'v'^* = p_k^D$. Now assume v^*v is minimal and central in $\bar{q}N_{i,j}(S_0)\bar{q}$, but v'^*v' is not minimal and central in $\bar{q}'N_{i,j}(S_0)\bar{q}'$, where $\bar{q} = vv^* + v^*v$, $\bar{q}' = v'v'^* + v'^*v'$, and S_0 as before. Then there must be some $w \in N_{i,j}(S_0)$ such that $ww^* = v^*v$ and $w^*w = v'^*v'$. Note since v^*v is minimal and central in $\bar{q}N_{i,j}(S_0)\bar{q}$, it is minimal and commutes with p_k^D in $N_{i,j}(S_0)$. Thus it must be in some factor summand of $N_{i,j}(S_0)$ which is a matrix algebra orthogonal to p_k^D . But using w , we see v'^*v' must be in this same factor summand, contradicting our assumption.

As in the proof of theorem 5.1 in [3] we see that $\bar{q}M(i, j)\bar{q} = vN(\bar{q}N_{i,j}(S_0)\bar{q} \cup \{v\})$, and furthermore that this is isomorphic to $\bar{q}N_{i,j}(S_0)\bar{q} *_{\mathbb{C} \oplus \mathbb{C}} M_2(\mathbb{C})$, where $\mathbb{C} \oplus \mathbb{C}$ is spanned by vv^* and v^*v and $M_2(\mathbb{C})$ is generated by v .

Then Lemma 4.2 from [3] shows $\bar{q}M(i, j)\bar{q}$ is isomorphic to an interpolated free group factor F (by our assumption on the size of projections in $A(i)$ we know this isn't hyperfinite or zero), possibly direct sum with some matrix algebras. Then F is contained in some free group factor summand F' in $M(i, j)$, and we assign k to $Q_{F'}$. From this we see that any free group factor in $M(i, j)$ other than F' is orthogonal to \bar{q} , and thus to p_k^D .

In fact, in this case if we know v^*v is not minimal and central in $\bar{q}N_{i,j}(S_0)\bar{q}$, then we know it is not minimal there. If it were minimal, but not central then there would be some partial isometry w also in $\bar{q}N_{i,j}(S_0)\bar{q}$, $w \in A$ so that $w^*w \leq v^*v$, and $ww^* \leq vv^*$, in which case, since v^*v is minimal, $w^*w = v^*v$, and thus (since the traces are the same) $ww^* = vv^*$, and so $p_k^D = ww^*$, and this is minimal in A , contradicting our assumption.

This means that there must be some value $\delta > 0$ so that $\tau(v^*v) - \tau(p) > \delta$ for all minimal projections $p \in \bar{q}N_{i,j}(S_0)\bar{q}$, $p \leq v^*v$. We may start our sequence $A_d(i)$ far enough along so that any minimal projection in $A_d(i)$ has trace less than δ . In this case we actually have $p_k^D A_d(i) p_k^D \subseteq F'$ where $k \in Q_{F'}$. Start our sequence far enough along that this is true for all k assigned with Method II.

We have now assigned all the elements of I_D that we are going to. We must now show that $Q_F \neq \emptyset$ for any factor summand F in $M(i, j)$ which is not orthogonal to $A_d(i)$ and $B_d(j)$.

To do this, take any $\ell \in I_{A_d(i)}$ or $I_{B_d(j)}$ (without loss of generality assume $I_{A_d(i)}$). Let $K = \{k \in I_D | p_k^D M_{n_\ell} p_k^D \neq 0\}$. If there is any $k \in K$ and factor summand F in $M(i, j)$ such that $k \in Q_F$, then $M_{n_\ell} \subseteq F$.

Conversely assume there is no such $k \in K$, i.e. $k \notin Q_F$ for all F . We will show that M_{n_ℓ} is then orthogonal to all free group factor summands of $M(i, j)$.

We know that for each $k \in K$, p_k^D is minimal in B , otherwise it would have been assigned by Method I. For each $k \in K$, where p_k^D is not central in B , choose a connector $v_k \in B_a$ such that $v_k v_k^* = p_k^D$, and let V be the set of these connectors. Now there cannot be any $v_k \in V$ such that $v_k v_k^* = p_k^D$ and $v_k^* v_k = p_{k'}^D$ for $k, k' \in K$, because choosing a connector $w \in M_{n_\ell} \subseteq A_d(i)$ so that $w^*w \leq p_k^D$ and $ww^* \leq p_{k'}^D$, we would have $v_k^* v_k$ not central in $\bar{q}N_{i,j}(S_0)\bar{q}$, meaning k would have been assigned by Method II, contrary to the hypothesis.

Thus we can choose S to be a set of connectors in $M(i, j)$ containing V , so that $N_{i,j}(S) = M(i, j)$ and such that for each $k \in K$ there is at most one $v \in S$ such that either vv^* or v^*v equals p_k^D .

Now then M_{n_ℓ} is a factor summand in $N_{i,j}(S \setminus V)$, since in this algebra every p_k^D not orthogonal to M_{n_ℓ} is minimal and central in $B(j)$. We now write $V = \{v_{k(1)}, \dots, v_{k(r)}\}$. We consider the chain of embeddings,

$$N_{i,j}(S \setminus V) \rightarrow N_{i,j}(S \setminus V \cup \{v_{k(1)}\}) \rightarrow \dots \rightarrow N_{i,j}(S) = M(i, j).$$

In each of them the algebra M_{n_ℓ} is embedded as the corner of a factor summand which is also a matrix algebra. Thus M_{n_ℓ} is orthogonal to all free group factor summands in $M(i, j)$.

If Method I or II is applied to some $k \in I_D$, in $M(i, j)$, it will also be applied in $M(i + 1, j)$. Furthermore if there is some factor summand F in $M(i, j)$ so that $k, k' \in Q_F$, then there must be a partial isometry $v \in M(i, j)$ so that $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$, and both vv^* and v^*v are in either $A_d(i)$ or $B_d(j)$ (though v may not be). Then, since that v is embedded in $M(i + 1, j)$, there must be some factor summand F' of $M(i + 1, j)$ so that $k, k' \in Q_{F'}$. Thus the Q_F must be eventually stable, and thus so must be the number of free group factor summands in the sequence of $M(i, j)$. Without loss of generality, by starting the sequence $A(i)$ and $B(j)$ far enough along, we assume the number of interpolated free group factor summands of $M(i, j)$ is constant in i, j , as is the family $\{Q_F\}_F$.

Consider the embedding $M(i, j) \rightarrow M(i + 1, j)$, which corresponds to applying a simple step to $A_d(i) \rightarrow A_d(i + 1)$. We will show that for any free group factor summand F in $M(i, j)$, with central support p_F , the restriction of the embedding $F \rightarrow p_F M(i + 1, j) p_F$ is either a standard embedding or the identity.

First assume it is the first type of simple step, where we make a copy of a matrix algebra M_{n_ℓ} , $\ell \in I_{A_d(i)}$. If there is a factor summand F with $k \in Q_F$ so that $p_k^D M_{n_\ell} p_k^D \neq 0$, then we know $M_{n_\ell} \subseteq F$. Then, as in the previous proposition, we can apply Lemma 3.3 to see that this is a standard embedding.

Note that if there is no such F , then M_{n_ℓ} is the corner of a matrix algebra factor summand of $M(i, j)$, in which case we end up with two copies of this in $M(i + 1, j)$.

Assume instead it is the second type of simple step, adding a connector between M_{n_ℓ} and $M_{n_{\ell'}}$, for some $\ell, \ell' \in I_{A_d(i)}$. If there are F, F' factor summands of $M(i, j)$ which contain M_{n_ℓ} and $M_{n_{\ell'}}$ respectively, then by our assumption $F = F'$, and as in the previous proposition we can apply Lemma 3.2, to show that $F \rightarrow p_F M(i + 1, j) p_F$ is standard.

If neither M_{n_ℓ} nor $M_{n_{\ell'}}$ is contained in an interpolated free group factor summand of $M(i, j)$, then these are corners of factor summands M_m and $M_{m'}$ in $M(i, j)$, which are embedded in a factor summand $M_{m'+m}$ in $M(i + 1, j)$.

Finally if only one of M_{n_ℓ} or $M_{n_{\ell'}}$ is contained in an interpolated free group factor summand, say M_{n_ℓ} , then we can apply Lemma 4.2 from [3] again to see that these are embedded in a free group factor summand F' of $M(i + 1, j)$ so that $F \rightarrow p_F F' p_F$ is standard or the identity, where p_F is the identity of F .

If $P_M(i, j)$ is the projection onto the hyperfinite and matrix portion of $M(i, j)$ we see that $P_M(i, j) \geq P_M(i', j')$ if $(i, j) \leq (i', j')$. Taking the inductive limit of those, we get a hyperfinite algebra.

As before the free dimensions of each $M(i, j)$ add up correctly, so $\text{fdim}(A *_D B) = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D)$. \square

Corollary 4.3. *If A and B are hyperfinite von Neumann algebras with finite dimensional abelian von Neumann subalgebra D , so that $G_D^{A,B}$ is connected, and so that any minimal projection p in A or B , with $p \leq p_k^D$ a minimal projection in D $\tau(p) < \frac{1}{2}\tau(p_k^D)$, then $A *_D B = L(F_s)$ $s = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D)$.*

Proof. Based on the condition $\tau(p) \leq \frac{1}{2}\tau(p_k^D)$, we know that there is no matrix algebra component. Furthermore, by the connectedness of $G_D^{A,B}$ there can be only one interpolated free group factor. \square

Definition 4.4. Define \mathcal{R}_2 to be the set of finite von Neumann algebras which are the direct sum of a finite number of interpolated free group factors and a hyperfinite von Neumann algebra. Note \mathcal{R}_2 strictly contains the set \mathcal{R} from [5].

Theorem 4.5. For $A, B \in \mathcal{R}_2$ and D a finite dimensional subalgebra of A and B , $M = A *_D B$ is in \mathcal{R}_2 , and if generating sets of A and B have free dimension d_A and d_B , respectively, then M has a generating set of free dimension $d_A + d_B - \text{fdim}(D)$.

Proof. As in Theorem 4.4 in [5], we proceed by induction. In this case on the total number of interpolated free group factor summands in A and B . The base case, where there are none, is Theorem 4.2.

From there we can proceed identically to the proof of Theorem 4.4 in [5]. Without loss of generality assume A contains at least one interpolated free group factor summand, and let p be the central support in A of this factor summand. Then by Lemma 4.3 in [3], we know $pMp \cong pA *_D (p\tilde{M}p)$, where $\tilde{M} = (pD \oplus (1-p)A) *_D B$. Then we note that $(pD \oplus (1-p)A)$ is in \mathcal{R}_2 and has one fewer interpolated free group factor summand than A . Thus, by our induction hypothesis $\tilde{M} \in \mathcal{R}_2$. Then since pA is an interpolated free group factor, we can apply Lemma 4.1 from [5] to see that pMp is an interpolated free group factor. Then, as in Theorem 4.4 of [5], we note $M \cong \tilde{M}(1 - C_M(p)) \oplus (C_M(p)M)$, and that $\tilde{M}(1 - C_M(p)) \in \mathcal{R}_2$ and $(C_M(p)M)$ is an interpolated free group factor. Thus $M \in \mathcal{R}_2$.

The free dimension calculation also proceeds exactly as in Theorem 4.4 of [5]. \square

5. EXAMPLES

Example 5.1. For one of the simplest examples, consider

$$M = L^\infty(\mu_1) \otimes R *_D^{\mathbb{C} \oplus_{1-\alpha} \mathbb{C}} L^\infty(\mu_2) \otimes R.$$

In this case, as long as the graph is connected, the embedding of D does not matter, nor do the measures μ_1 and μ_2 . Since there are no minimal projections in A or B , we know that M is a single interpolated free group factor. We also know $\text{fdim}(M) = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D) = 1 + 1 - (1 - \alpha^2 - (1 - \alpha)^2)$. Thus $M = L(F_{1+\alpha^2+(1-\alpha)^2})$.

Example 5.2. Next we look at an example where we get the direct sum of two free group factors, despite having connected graph. Consider:

$$\begin{aligned} A &= \overset{p}{R} \oplus \underset{1/8}{M_2} \oplus \underset{1/8}{M_2} \oplus \overset{q}{R} \\ B &= \underset{1/8}{\mathbb{C}} \oplus \underset{1/8}{M_2} \oplus \underset{1/4}{\mathbb{C}} \oplus \underset{1/8}{M_2} \oplus \underset{1/8}{\mathbb{C}} \\ D &= \overset{p_1}{\mathbb{C}} \oplus \overset{p_2}{\mathbb{C}} \oplus \overset{p_3}{\mathbb{C}} \oplus \overset{p_4}{\mathbb{C}} \oplus \overset{p_5}{\mathbb{C}} \\ &\quad \underset{1/4}{\quad} \underset{1/8}{\quad} \underset{1/4}{\quad} \underset{1/8}{\quad} \underset{1/4}{\quad} \end{aligned}$$

Where $\tau(p) = \tau(q) = 1/4$.

$$\begin{aligned} p_1 &= \left(I_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_R \right) \in A, \left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \in B \\ p_2 &= \left(0_R, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_R \right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \in B \\ p_3 &= \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0_R \right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \in B \\ p_4 &= \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0_R \right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \in B \\ p_5 &= \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, I_R \right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 1 \right) \in B \end{aligned}$$

We use $A(i) = M_i \oplus M_2 \oplus M_2 \oplus M_i$ and $B = B_a = B(j), \forall j$. Applying theorem 5.1 from [3], we see $M(i, j) = L(F_{\frac{9}{8} - \frac{1}{4i^2}}) \oplus L(F_{\frac{9}{8} - \frac{1}{4i^2}})$. Thus as $i \rightarrow \infty$, $M = A *_D B = L(F_{\frac{9}{8}}) \oplus L(F_{\frac{9}{8}})$. Checking, we see $\text{fdim}(A) = \frac{31}{32}$, $\text{fdim}(B) = \frac{7}{8}$ and $\text{fdim}(D) = \frac{25}{32}$, this gives us that $\text{fdim}(M) = \frac{17}{16}$ which matches our result.

Example 5.3. Next consider the case, $A = L^\infty[0, 1] \oplus M_2$ and $B = M_2 \oplus \mathbb{C}$, and

$$D = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}. \quad p_1 = (I_{L^\infty}, 0) \in A \text{ and } (e_{11}, 0) \in B, \quad p_2 = (0, e_{11}) \in A \text{ and } (e_{22}, 0) \in B, \text{ and } p_3 = (0, e_{22}) \in A, \text{ and } (0, 1) \in B.$$

Then we set $A(i) = \mathbb{C}^i \oplus M_2$, and $B(j) = B \forall j$. Then $M(i, j) = \mathbb{C}^i \otimes M_3$. Taking the inductive limit, this is $L^\infty \otimes M_3$.

Example 5.4. Similarly, if $A = (L^\infty[0, 1] \otimes R) \oplus M_2$ and $B = M_2 \oplus \mathbb{C}$, and $D =$

$$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}. \quad p_1 = (I_{L^\infty \otimes R}, 0) \in A \text{ and } (e_{11}, 0) \in B, \quad p_2 = (0, e_{11}) \in A \text{ and } (e_{22}, 0) \in B, \text{ and } p_3 = (0, e_{22}) \in A, \text{ and } (0, 1) \in B.$$

Then we set $A(i) = \mathbb{C}^i \otimes M_i \oplus M_2$, and $B(j) = B \forall j$. Then $M(i, j) = \mathbb{C}^i \otimes M_{3i}$. Taking the inductive limit, this is $L^\infty \otimes R$.

Example 5.5. Let $A = R$ and $B = M_2 \oplus \mathbb{C}$ and $D = \mathbb{C} \oplus \mathbb{C}$, with α irrational.

Then we can set $A(i) = M_{2^{i-1}} \oplus \mathbb{C}^a \oplus \mathbb{C}^b$, where $a + b = 1/2^i$, $q_1 \leq p_1$ and $q_2 \leq p_2$. $B(j) = B$ for all j . Then $M(i, j) = L(F_s) \oplus \mathbb{C}$ for some s . Noting $b \rightarrow 0$ as $i \rightarrow \infty$, so $M = L(F_s)$, where s can be worked out from the free dimension $(1 + (3\alpha)/4)$.

Example 5.6. Let $A = M_2 \oplus \bigoplus_{i=1}^\infty M_2$ and $B = M_2 \oplus \mathbb{C}$, and $D = \mathbb{C} \oplus \mathbb{C}$, where $p_i \leq q_2$ for all $i \geq 1$, $q_1 \leq p_0$, and $\tau(p_0 q_2) = 1/3$. Then $M = A *_D B$ is the

amalgamated free product of two multimatrix algebras, with $G_D^{A,B}$ connected and D finite dimensional. The free product is:

$$M_4 \oplus_{1/6} \bigoplus_{i=1}^{\infty} M_2 \oplus_{\frac{1}{6 \cdot 2^i}}.$$

The above example contradicts the last statement of Theorem 5.1 of [3]. That statement is erroneous and should be modified by changing “ J and X are finite” to “ J is finite”.

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